Stress and Strain in Solids

A solid is stressed by applying external forces such that both the net force and the net torque are zero, i.e. static equilibrium.

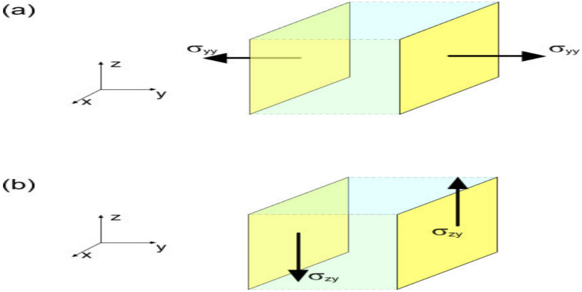


Figure 1: (a) Zero net force and (b) zero net torque, applied to a cubic volume element of face area A. The first suffix indicates the direction in which the stress (force) acts and the second suffix indicates the direction of the normal to the face upon which the stress acts.

The Stress Tensor I

A force $\mathbf{F}_x$ applied in the x-direction to a plane with area A_x whose normal is also in the x-direction produces a stress component:

$$\sigma_{xx} = \frac{F_x}{A_x}$$

Likewise, the same force applied to a plane whose normal is in the y-direction produces the stress component:

$$\sigma_{xy} = \frac{F_x}{A_y}$$

Since stresses on opposite faces of the cube are equal and opposite we only need to consider three faces. Thus, stress in general is described by a second-rank tensor σ with nine components $\sigma_{ij}(i, j = x, y, z)$, conveniently written as a 3×3 array:

The Stress Tensor II

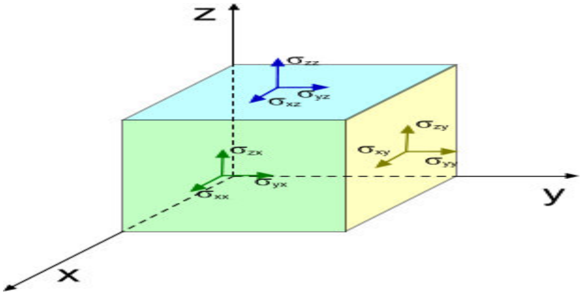


Figure 2: Representation of the 'three face stresses'.

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Special cases I

However, only six of these stress components are independent because of the constraint of zero torque, which implies that:

$$\sigma_{xy} = \sigma_{yx}; \sigma_{yz} = \sigma_{zy}; \sigma_{zx} = \sigma_{xz}$$

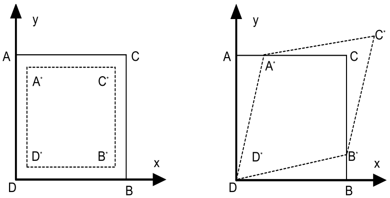


Figure 3: Uniform hydrostatic compressive stress and pure shear deformation (at constant volume).

Special cases II

Special cases of stress have most of the six independent stress components equal to zero. For instance for **uniaxial stress** in the x-direction, the only non-zero component is $\sigma_{xx} = \sigma$.
For **hydrostatic stress**, the non-zero components are the diagonal components

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P$$

For **pure shear**, in which the shape but not the volume of the solid is changed, is represented by say

$$\sigma_{xy} = \sigma_{yx} = \sigma$$

being the only non-zero components.

Diagonalization I

A particular orientation of the spatial coordinates can always be found such that the general stress matrix has diagonal components only, i.e.

$$\sigma' = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$$

the six independent variables defining the stress system now being the three principal stresses $\sigma_i (i = x, y, z)$ acting along the principle axes, and the three variables needed to determine the orientation of the principle axes with respect to the original coordinate system.

Matrix Transformation I

A general stress matrix, transformed to the principal axis system can always be written as

$$\begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix} + \begin{bmatrix} (\sigma_x - \sigma_0) & 0 & 0 \\ 0 & (\sigma_y - \sigma_0) & 0 \\ 0 & 0 & (\sigma_z - \sigma_0) \end{bmatrix}$$

where the new stress component is given by

$$\sigma_0 = (\sigma_x + \sigma_y + \sigma_z)/3$$

The first term of the transformed stress matrix represents a purely hydrostatic term, with

$$\sigma_0 = -P = (\sigma_x + \sigma_y + \sigma_z)/3$$

which causes a change in volume (dilatoric), but not of shape, of an elastically isotropic solid.

Matrix Transformation II

The second term of the transformed stress matrix represents a pure shear, causing a change of shape (deviatoric), but not of volume, of a solid, since the sum of the diagonal components equals zero, i.e. $\sum_{x,y,z} \sigma_{ii} = 0$

Non uniform deformation: Strains

The application of an external force to a solid causes a deformation because different points in the material are displaced by different amounts. Consider a point P initially at $\mathbf{r}$ and a neighbouring point Q initially at $\mathbf{r} + \mathbf{u}$, displaced under the action of stress to $P^{'}$ at $(\mathbf{r} + \Delta\mathbf{r})$ and $Q^{'}$ at $(\mathbf{r} + \Delta\mathbf{r} + \mathbf{u} + \Delta\mathbf{u})$ respectively.

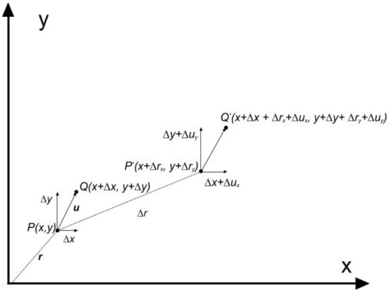


Figure 4: 2D representation of infinitesimal homogeneous elastic strain.

Strain Components I

For small relative displacements ($|\Delta \mathbf{u}| << |\Delta \mathbf{r}|$), the components of the relative displacement are given by

$$\Delta u_i = \frac{\partial u_i}{\partial x}\Delta x + \frac{\partial u_i}{\partial y}\Delta y + \frac{\partial u_i}{\partial z}\Delta z$$

for $i = x, y, z$. The strain components $e_{ij} \equiv \varepsilon_{ij}$ are then defined in terms of the dimensionless displacement gradients as

$$e_{ii} = \frac{\partial u_i}{\partial i} \; (i = x, y, z)$$

and

$$e_{ij} = e_{ji} = \left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) \; (i = x, y, z)$$

Therefore, the strain components can, like the stress components, be written as a (3×3) matrix (symmetric) with again only six of the nine components

Strain Components II

being independent because of the requirement that the off-diagonal components obey the condition $e_{ij} = e_{ji}$ in order to exclude rigid rotations. Special cases of strain include uniaxial strain in the x-direction with only one non-zero strain component being $e_{xx} = e$; uniform dilation or compression, resulting from hydrostatic stress, with $e_{ii} = e \, (i = x, y, z)$; and pure shear with $e_{ij} = e_{ji} = e \, (i = x, j = y)$ or other orthogonal pair.

General Strain

As with a general stress, a general strain can be separated into a dilation-strain component (volume change) and a pure strain (deviatoric-strain) component (shape change). For a general strain component

$$e_{ij} = \frac{e_0}{3}\delta_{ij} + (e_{ij} - \frac{e_0}{3}\delta_{ij})$$

where the Kronecker delta symbol has the properties:

$$\delta_{ij} = 1, \quad i = j$$
$$\delta_{ij} = 0, \quad i \neq j$$

Notation simplification

Since there are only six independent stress and strain components, a convenient short-hand notation is to relabel the component indices as follows:

$$
\begin{aligned}
xx &\rightarrow \quad 1 \\
yy &\rightarrow \quad 2 \\
zz &\rightarrow \quad 3 \\
yz &\rightarrow \quad 4 \\
zx &\rightarrow \quad 5 \\
xy &\rightarrow \quad 6
\end{aligned}
$$

Hooke's law states that the stress and strain are directly proportional to each other. Thus, in terms of elastic-stiffness coefficients, C_{ij}, a stress coefficient can be written as a function of a strain as

$$\sigma_i = \sum_{j=1}^{6} C_{ij} e_j$$

Stress-strain coefficients

The relationship between stress and strain coefficients can also be written in matrix form as

$$\left(\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \hline \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{array}\right) = \left(\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ \hline C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{array}\right) \left(\begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \hline e_4 \\ e_5 \\ e_6 \end{array}\right)$$

Compliance - a useful alternative! I

Alternatively, a strain coefficient can be expressed in terms of the elastic-compliance coefficients, S_{ij}, as a function of the stress:

$$e_i = \sum_{j=1}^{6} S_{ij}\sigma_j$$

where both elastic compliance and stiffness are quantities describing a material as an elastic continuum.

Compliance - a useful alternative! II

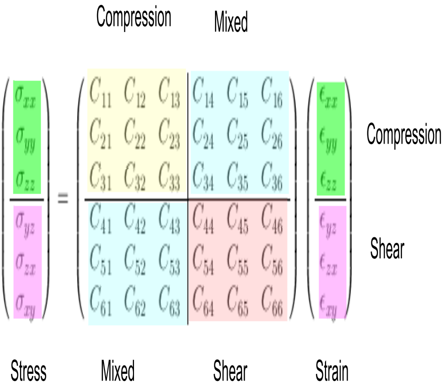


Figure 5: Stress Strain Matrix

Physical Meaning

Physical meaning of e_{ij}

$$\begin{aligned} e_{11} &= \frac{\partial u_x}{\partial x} \\ e_{22} &= \frac{\partial u_y}{\partial y} \\ e_{33} &= \frac{\partial u_z}{\partial z} \end{aligned}$$

Simple tensile strains along the x, y, z axes are positive for tension and negative for compression. Deformation in the xy plane of a rectangular thin film corresponds to:

$$e_{12} \neq 0; e_{21} \neq 0$$

$$e_{11} = e_{22} = 0$$

Pure Shear

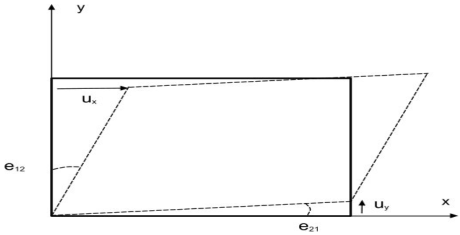


Figure 6: Pure Shear

With angles greatly exaggerated, this corresponds to:

$$e_{12} \approx \frac{\partial u_x}{\partial y}$$

$$e_{21} \approx \frac{\partial u_y}{\partial x}$$

which equals the angles respectively.

Pure Rotation

Possible combinations of e_{12} and e_{21}:

- Pure Shear $e_{12} = e_{21}$
- Pure Rotation $e_{12} = -e_{21}$
- Simple Shear $e_{12} \neq 0; e_{21} = 0$

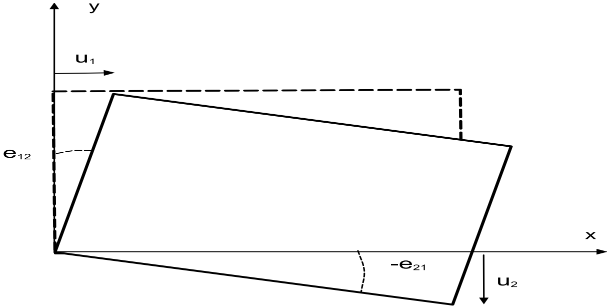


Figure 7: A pure rotation.

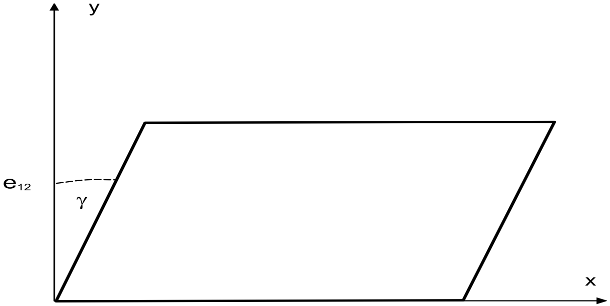
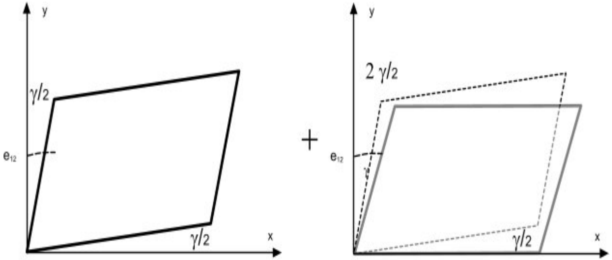


Figure 8: Simple Shear



Generalization I

$$e = \begin{pmatrix} 0 & \gamma/2 & 0 \\ \gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma/2 & 0 \\ -\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_{ij} = \varepsilon_{ij} + \omega_{ij}$$

$$\varepsilon_{ij} = 1/2(e_{ij} + e_{ji})$$

$$\omega_{ij} = 1/2(e_{ij} - e_{ji})$$

$e_{ij} = e_{ji}$ - symmetric with respect to i and j: measures shape change!
$\omega_{ij} = -\omega_{ji}$ - antisymmetric with respect to i and j: measures rotation.
$\omega_{ij} = 0(i = j)$
Therefore ω_{ij} has three independent elements implying that three independent rotations are possible, one about each axis.

Simplification of the Matrix I

Of the 36 elastic stiffness (or compliance) coefficients in the most general case of triclinic crystals, 21 are independent and non zero as a result of the general condition

$$C_{ij} = C_{ji}$$

The presence of higher symmetry reduces the number of coefficients further. For cubic crystals, just three components are independent, C_{11}, C_{12} and C_{44} with

$$\begin{aligned} C_{11} &= C_{22} = C_{33}, \\ C_{12} &= C_{13} = C_{23}, \\ C_{44} &= C_{55} = C_{66}, \end{aligned}$$

and all other coefficients being zero.

Matrix: Simple Cubic Crystal

$$
\left(\begin{array}{ccc|ccc}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}
\end{array}\right)
$$

Stiffness and Compliance Coefficients

These elastic stiffness constants are related to the corresponding elastic compliance coefficients via the relationships

$$S_{11} = \frac{C_{11} + C_{12}}{(C_{11} - C_{12})(C_{11} + 2C_{12})} > 0,$$

$$S_{12} = \frac{-C_{12}}{(C_{11} - C_{12})(C_{11} + 2C_{12})} \leq 0,$$

$$S_{44} = \frac{1}{C_{44}}$$

as obtained from inversion of the stiffness matrix.

Isotropic Solids

In the case of elastically isotropic solids (amorphous solids), the number of independent elastic coefficients decreases to two since, in addition to the equalities applicable to cubic materials, the coefficients are further linearly related to each other by the equation

$$C_{11} = C_{12} + 2C_{44}$$

Lamé Constants: For such isotropic materials, the two independent elastic stiffness coefficients are conventionally referred to as the Lamé constants:

$$\lambda \equiv C_{12}$$

and

$$\mu = C_{44}$$

with

$$C_{11} \equiv (\lambda + 2\mu)$$

Uniaxial Stress: Isotropic Media I

Uniaxial Stress: For uniaxial stress,Young's modulus is defined as

$$E = \frac{\sigma_{xx}}{e_{xx}}$$

and so

$$E = \frac{1}{S_{11}} = \frac{(C_{11} - C_{12})(C_{11} + 2C_{12})}{C_{11} + C_{12}} = \mu\frac{(3\lambda + 2\mu)}{(\lambda + \mu)}$$

for the case of isotropic media. Under such stress loading, the sample also deforms in directions perpendicular to the stress direction. This behaviour is quantified by Poisson's ratio, defined as

$$\nu = \frac{|\text{transverse strain}|}{\text{normal strain}} = \frac{|e_{yy}|}{e_{xx}}$$

with

$$\nu = \frac{-S_{12}}{S_{11}} = \frac{C_{12}}{(C_{11} + C_{12})} = \frac{\lambda}{2(\lambda + \mu)}$$

Uniaxial Stress: Isotropic Media II

where $0 \leq \nu \leq 0.5$.

Hydrostatic Stress

Hydrostatic Stress: In the case of hydrostatic stress (pressure), the bulk modulus B (equal to the inverse of the compressibility, $\kappa = \frac{1}{B}$ relates the pressure to the dilation (fractional change in volume) e_0,

$$B = \frac{P}{e_0}$$

with

$$B = \frac{1}{3(S_{11} + 2S_{12})} = \frac{(C_{11} + 2C_{12})}{3} = \frac{(3\lambda + 2\mu)}{3}$$

The bulk modulus is connected to microscopic quantities relating to the interatomic potential.

Pure Shear

Pure Shear: Finally, for the case of pure shear, the shear (or rigidity) modulus is defined as:

$$G = \frac{\text{shear stress}}{\text{shear strain}} = \frac{\sigma_{xy}}{e_{xy}}$$

giving for isotropic solids

$$G = \frac{1}{2(S_{11} - S_{12})} = \frac{(C_{11} - C_{12})}{2} \equiv C_{44} = \mu$$

Some Interrelationships

Some Interrelationships: The various elastic moduli are inter-related; for example

$$G = \frac{E}{2(1+\nu}$$

and

$$B = \frac{E}{3(1-2\nu)}$$

Deformation Modes of Cubic Crystals 1

(a) Dilation by hydrostatic stress.
Hydrostatic: Under a hydrostatic pressure P:

$$\sigma_1 = \sigma_2 = \sigma_3 = -P$$

$$\sigma_4 = \sigma_5 = \sigma_6 = -0$$

and the dilation $\triangle = e_1 + e_2 + e_3$ is

$$\triangle = e_1 + e_2 + e_3$$

without rotation. Using these relationships we can show that the bulk modulus B is given by:

$$B = \frac{-p}{\triangle} = \frac{1}{3}(C_{11} + 2C_{12})$$

(b) Shear on a cube face parallel to a cube axis.

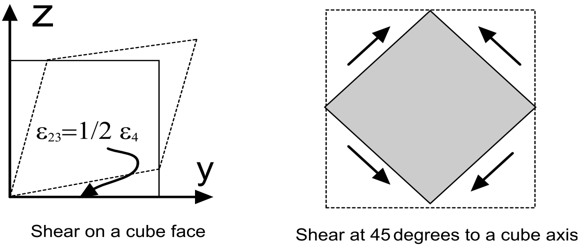


Figure 10: Two shear modes for a cubic crystal.

For example on the (010) plane in the [001] direction:

$$\sigma_4 = C_{44} e_4$$

C_{44} is a shear modulus, which by convention we name μ_0.

(c) Shear at 45° to a cube axis
For example, on the (110) plane in the $[1\bar{1}0]$ direction. In this case the shear modulus is:

$$\mu_1 = \frac{1}{2}(C_{11} - C_{12})$$

This allows us define an anisotropy factor,

$$\frac{\mu_0}{\mu_1} = \frac{2C_{44}}{(C_{11} - C_{12})}$$

which of course will be one for an isotropic material.
In terms of the Lamé constants:

$$\lambda = C_{12}; \mu = C_{44}; \lambda + 2\mu = C_{11}$$

which explains why cubic crystals are not in general isotropic.

Deformation Modes of Cubic Crystals 2 II

	W	Al	Fe	Cu	Na
$\frac{\mu_0}{\mu_1}$	1.0	1.2	2.4	3.2	7.5
Structure	bcc	fcc	bcc	fcc	bcc